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*ECONOMIC DYNAMICS MODELS: THEORY, APPLICATIONS, COMPUTER AIDED IMPLEMENTATION**

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In the paper, a survey of theoretical and applied results obtained in the framework of the scientific school at the Department of Information Systems and Mathematical Methods in Economics, Perm State University, is given. It covers the period from 2008 to 2015. The theoretical results are based on the principal statements of the contemporary theory of functional differential equations worked out by the participants of the Perm Seminar under the leadership of Prof. N.V. Azbelev (1922–2006). The focus of attention is on problems of forecasting, boundary value problems (problems of attainability), control problems, and problems of stability for the dynamic models that allow one to take into account aftereffects and effects of impulse disturbances (shocks). For the mentioned problems, sufficient conditions of the solvability are obtained, methods of constructing program controls and the corresponding trajectories are proposed. Algorithms of the computer-assisted study of the control problems are worked out, including algorithms of correction for certain ill-posed problems. The applied results use the achievements of the theory and are implemented in the form of software tools for the study and solution of the real economy problems such as forecasting, control and stability analysis as applied to models of socio-economic development of the regions of the Russian Federation and the Russian Economy as a whole.

Keywords: economic dynamic models, forecasting problems, boundary value problems, control problems, information analytical systems, decision making systems, business intelligence systems.

Introduction

Here we give a survey of theoretical and applied results obtained in the framework of the scientific school at the Department of Information Systems and Mathematical Methods in Economics, Perm State University, that covers the period from 2008 to 2015. The earlier works are presented in the paper [2; 3] devoted to the 50th anniversary of Faculty of Economics, Perm State University, and in the monographs [1; 46]. The theoretical results are based on the principal statements of the contemporary theory of functional differential equations worked out by the participants of the Perm Seminar under the leadership of Prof. N.V. Azbelev (1922–2006). The focus of attention is on problems of forecasting, boundary value problems (problems of attainability), control problems, and problems of stability for the dynamic models that allow to take into account aftereffects and effects of impulse disturbances (shocks). For the mentioned problems, sufficient conditions of the solvability are obtained, methods of constructing program controls and the corresponding trajectories are proposed. Algorithms of the computer-assisted study of the control problems are worked out,

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The applied results use the achievements of the theory and are implemented in the form of software tools for the study and solution of the real economy problems such as forecasting, control and stability analysis as applied to models of socio-economic development of the regions of the Russian Federation and the Russian Economy as a whole.

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1. Control problems

Dynamic models under consideration cover a wide class of models arising under studying real-world economic and ecology-economic processes with taking into account impulse actions (considered as elements of control), impulse external disturbances, and aftereffect (time delay). Impulse actions result in jump-like changes in the system state and lead to introducing discontinuous solutions of differential equations with ordinary derivative.

These equations are considered in the space $DS(m)$ that is a finite dimensional extension of the traditional space of absolutely continuous functions (see below). Such an approach to the systems with jumps was proposed in [5]. It doesn’t use the complicated theory of generalized functions (i.e. distributions) and finds many applications. Conditions of the solvability to the control problems for linear functional differential systems with trajectories from $DS(m)$ as well as constructive methods and algorithms of constructing program controls are presented in [31; 32; 36; 37]. Therewith possible jumps of trajectories are considered as components of control actions in combination with the traditional control from the space L_2, and the aim of control is defined as the attainment of a prescribed value by each of the given linear functionals whose number in total is not equal to the dimension of the system. The latter circumstance and the general form of the on-target functionals are used in [30] to hold a trajectory in a neighborhood of a given normative trajectory during a given period of time. Some possible effects arising due to the use of impulse controls jointly with controls from L_2 are discussed in [31], where it is shown, in particular, that the use of impulse control can reduce the total cost of the given goals attainment.

Here we follow the notation and the principal statements of the theory of functional differential equations in its part concerned with linear impulsive systems [15, p. 123–130] (see also [16, p. 124–134]; [17, p. 100–108]). Denote by $L = L[0,T]$ the space of Lebesgue summable functions $z:[0,T]\to R^n$ with the norm $\|z\|_{L_n} = \int_0^T |z(s)|_n\,ds$, where $|\cdot|_n$ stands for a norm in R^n (in the sequel we shall omit the index *n* if the dimension of the space is obvious). To describe the trajectories with jumps of the first kind at the points $t_1 < t_2 < ... < t_m < T$ $(t_1 > 0)$, we follow [5] and introduce the space $DS(m)$ of piece-wise absolutely continuous functions $x:[0,T]\to R^n$ of the form

$$x(t) = \int_0^t z(s)ds + x(0) + \sum_{k=1}^{m} \chi_{[t_k,T]}(t)\Delta x(t_k), \qquad (1)$$

where $z \in L_n$, $\Delta x(t_k) \equiv x(t_k) - x(t_k - 0)$ and $\chi_{[t_k,T]}(t)$ is the characteristic function of the segment $[t_k,T]$. The norm in $DS(m)$ is defined by the equality

$$\|x\|_{DS(m)} = \|\dot{x}\|_L + |x(0)|_n + \sum_{k=1}^{m} |\Delta x(t_k)|_n .$$

Next we denote by $AC[0,T]$ the space of absolutely continuous $x:[0,T]\to R^n$ with the norm $\|x\|_{AC} = \|\dot{x}\|_L + |x(0)|_n$. Thus $DS(m)$ is a finite-dimensional extension of $AC[0,T]$.

To describe the system under control, we introduce the linear operator $\mathcal{L}$:

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int_0^t K(t,s)\dot{x}(s)ds + A(t,0)x(0) . \qquad (2)$$

Here the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $\{(t,s): 0 \le s \le t \le T\}$ and such that the estimates

$$|k_{ij}(t,s)| \le \kappa(t), \quad i,j = 1,...,n,$$

hold on this set with a κ summable on $[0,T]$, and the elements of $(n\times n)$-matrix $A(t)$ are summable on $[0,T]$ too. The operator $\mathcal{L}: DS(m) \to L$ is bounded. Functional differential system $\mathcal{L}y = f$ covers differential equations with concentrated and/or distributed time delay and Volterra integro-differential systems.

The space of all solutions to the homogeneous system $(\mathcal{L}x)(t) = 0,\ t\in[0,T]$, is of dimension $n+mn$. Let $\{x_1,...,x_{n+mn}\}$ be a basis in this space. The matrix $X = \{x_1,...,x_{n+mn}\}$ is called the fundamental matrix (we assume, for definiteness, that $rX = E$). The so-called principal boundary value problem $\mathcal{L}x = f, rx = \sigma$ is uniquely solvable for any $f \in L, \sigma \in R^{n+mn}$, and its solution has the representation

$$x(t) = X(t)\sigma + \int_0^t C(t,s)f(s)ds , \qquad (3)$$

where $C(t,s)$ is the Cauchy matrix.

Let $\ell: DS(m) \to R^N$ be the linear bounded functional. There takes place the representation

$$\ell x = \int_0^T \Phi(s)\dot{x}(s)ds + \Psi_0 x(0) + \sum_{k=1}^{m} \Psi_k \Delta x(t_k),$$

where elements of measurable ($N \times n$)-matrix are bounded in essence, and Ψ_k, $k = 0,...,m,$ are ($N \times n$)-matrices with real-valued elements.

Consider the control problem

$$\mathcal{L}x = Fu + f, \quad x(0) = \alpha, \quad \ell x = \beta. \qquad (4)$$

Here $F : L_2 \to L$ is a linear bounded operator, L_2 is the space of square summable functions $u:[0,T] \to R^r$ with the inner product $\langle u,v \rangle = \int_0^T u^{\mathrm{T}}(t)v(t)\,dt$ ($\cdot^{\mathrm{T}}$ stands for transposition). The goal of control in (4) is given with the use of a vector-functional $\ell : DS(m) \to R^N$: it have to take the vector value β on a trajectory of the system $\mathcal{L}x = Fu + f$ under a control u.

In this survey, we restrict ourselves to one completely formulated main theorem that gives a necessary and sufficient condition of the solvability to problem (4). As for the rest, we refer the reader to the corresponding papers and give only brief comments.

To formulate the theorem, we introduce the following designations:

$$\Theta(s) = \Phi(s) + \int_s^T \Phi(\tau)\, C'_\tau(\tau,s)\, d\tau,$$

$$\Xi = \int_0^T \Phi(s)\dot{X}(s)\, ds = (\Xi_1 \,|\, \Xi_2),$$

where Ξ_1 is the $(N \times n)$-matrix whose columns are first n columns of $N \times (n + mn)$-matrix Ξ ;

$$M = \int_0^T [F^*\Theta](s)[F^*\Theta]^{\mathrm{T}}(s)\, ds,$$

here $F^* : L^* \to L^*_2$ is the adjoint operator to F.

Theorem 1 ([32]). *The problem* (4) *is solvable if and only if the linear algebraic system*

$$\begin{aligned}[\Xi_2 + (\Psi_1,...,\Psi_m)]\cdot\lambda + M\cdot\mu = \\ = \beta - \int_0^T \Theta(s) f(s)\, ds - (\Xi_1 + \Psi_0)\cdot\alpha\end{aligned} \qquad (5)$$

is solvable with respect to $(nm+N)$*-vector* $col(\lambda, \mu)$*. Every solution* $col(\lambda_0, \mu_0)$*,*

$\lambda_0 = col(\lambda_0^1,...,\lambda_0^m)$*, of the system* (5) *determines the control that solves the problem* (4) :

$$\Delta x(t_k) = \lambda_0^k,\ k = 1,...,m, \quad u(t) = [F^*\Theta]^{\mathrm{T}}(t)\cdot\mu_0.$$

Let us give some explanations how one could use this theorem to hold a system in a given neighborhood of the normative trajectory. Without loss of generality we consider the case of the zero normative position. Thus it is sufficient to hold a system in a neighborhood of the origin. Let us fix a $T_1 \in (0,T)$ and first solve the control problem

$$\mathcal{L}x = Fu + f,\ t \in [0,T],\ \ x(0) = \alpha,\ x(T_1) = 0. \quad (6)$$

At the point $t = T_1$, the system takes the right position. If take off the control at this moment, that is put $u(t) = 0,\ t \in [T_1,T]$, then, even for the case of $f(t) \equiv 0$, the system with aftereffect will lost the zero position as it has in general a nonzero prehistory which plays the role of disturbance. In order to hold the system in a neighborhood of zero, we use the following additional conditions. Let us add to the conditions of (6) the equalities

$$\int_{T_1}^T V_j(s)\, x(s)\, ds = 0,\ j = 1,2,...,\nu. \qquad (7)$$

Here $V_j = diag(v_j,...,v_j)$; $v_1,...,v_\nu,...$ is a linearly independent system of elements from $L_2[T_1, T]$ such that their linear span is everywhere dense in this space. Under some natural assumptions, for any given radius of the ball in $L_2[T_1, T]$, centered by the origin, there exists a ν such that conditions (7) provide us with the property that the corresponding trajectory x belongs on the segment $[T_1, T]$ to the above mentioned neighborhood.

In [31] the case is considered when the matrix M in (5) is nonsingular. In this case, problem (5) is solvable in the class of controls $u \in L_2$ for any collection of impulse actions, and impulses can be used to minimize the total cost of control. Let us note that in economic dynamic problems, impulse control is based on the possibility of change of the system state instantly at certain time moments due to the corresponding investments as an addition to a regular financing. As it takes place, estimating the total cost, we can take into account concrete parameters of financial program, say, parameters and conditions of credits. As is shown in [30], by virtue of instant financial actions one can reduce the total cost of the given goals attainment.

In [38], a closely related question is discussed, namely, the question of the dependence the total cost of control on a time delay of the control implementation. An approach to the problem of optimal delay is proposed.

In [4; 40; 42; 43], for systems with discrete time, the problem of correction of inconsistent control problems is considered. Two kinds of correction are under study, namely, the structural one and the resource correction. The algorithms of the correction are based on the results of I.I. Eremin and his collaborators [23]. It should be noted that the situation of inconsistency (ill-posedness) is met with quite often in practice of the study of real-world economic problems [4] .

Dynamic models considered in [21; 34; 35] are concrete realizations of the so-called abstract functional differential equation (AFDE). Theory of AFDE is thoroughly treated in [16; 17]. On the other hand, the systems under consideration are very typical ones met with in mathematical modeling economic dynamic processes and covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference) and with impulsive perturbations resulting in system's state jumps at prescribed time moments. The equations of the system contain simultaneously terms depending on continuous time, $t \in [0,T]$ and discrete, $t \in \{0, t_1, ..., t_N, T\}$, this is why the term "hybrid" seems to be suitable. As this term is deeply embedded in the literature in different senses, we follow the authors used the more definite name "continuous-discrete systems" (CDS). For the considered CDS's, in [21; 36; 37] the question on the solution representation is solved, the conditions of the solvability of boundary value problems and control problems are obtained in the form which is used under computer-assisted studying these problems. In [36; 37] the main result is a detailed description of all controls that solve the control problem by the controls generated by the discrete subsystem. The questions of optimal correction applied to inconsistent hybrid control problems are studied in [40; 41].

In [33] the system under consideration is subject to impulse disturbances which result in trajectory jumps. It is assumed that neither initial moments nor values of jumps are known in advance. A construction of regular (not impulse) control is proposed, which solves the control problem with a given set of objective functionals, despite of the action of impulses. It assumed also that the information about performed jumps is available to the beginning of the action of correcting controls, which are positional with respect to jumps of the realized trajectory. For the successive compensation of occurring jumps, a feedback (additional summands in motion equations) is introduced. An example given in this paper demonstrates that in the case of ignoring the proposed procedure the solution of the control problem is more expensive (needs a greater resource).

In [35] the boundary value problems

$$\mathcal{L}x = f, \quad \ell x = \beta$$

for functional differential systems are considered when the number of boundary conditions is greater than the dimension of the system in the case of approximate fulfilment of boundary conditions:

$$\mathcal{L}x = f, \quad |\ell x - \beta|_N \le \varepsilon. \tag{8}$$

The boundary value problems (8) are connected with studying the problems on the attainability for given indexes of development to the economic system under consideration. The approach is based on theorems whose conditions allow one to check up them by special reliable computing procedures. Dynamic models under consideration cover many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference).

2. Problems of Stability

The recent general theory of functional differential equations [15; 16; 17] allowed us to give a clear and concise description of their basic properties including the properties of solution stability. At the same time broad classes of linear hybrid functional differential systems with after-effect (LHFDSA) arising in many applications are not formally covered by the developed theory and remain out of view of specialists using functional differential and difference systems with after-effect for simulation of real processes. Below we suggest hybrid functional differential analogues of fundamental assertions of the theory of functional differential equations for problems of stability.

2.1. First, let us consider the case when one of the equations is a linear differential one and is defined on a set of discrete points, and the other one is a linear functional differential equation with aftereffect (LFDEA) on a semiaxis. For this case we describe the W-method scheme of N.V. Azbelev

Let us denote the infinite matrix with the columns $y(-1), y(0), y(1), ..., y(N), \ldots$ of size n, by $y = \{y(-1), y(0), y(1), ..., y(N), \ldots\}$ and the infinite matrix with columns $g(0), g(1), ..., g(N), \ldots$ the of size n. by $g = \{g(0), g(1), ..., g(N), \ldots\}$.

Each infinite matrix

$$y = \{y(-1), y(0), y(1), ..., y(N), \ldots\}$$

can be associated with the vector function

$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + \\ + y(1)\chi_{[1,2)}(t) + ... + y(N)\chi_{[N,N+1)}(t) + \ldots$$

Similarly, each of the infinite matrices $g = \{g(0), g(1), ..., g(N), \ldots\}$ can be associated with the vector function

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + ... \\ ... + g(N)\chi_{[N,N+1)}(t) + \ldots$$

Let us denote the vector function $y(t) = y([t]),\ t \in [-1, \infty)$, by $y(t) = y[t]$ and the vector function $g(t) = g([t]),\ t \in [0, \infty)$, by $g[t]$.

The set of vector functions $y[\cdot]$ is denoted by ℓ_0. The set of vector functions $g[\cdot]$ is denoted by ℓ. Let $(\Delta y)(t) = y(t) - y(t-1) = y[t] - y[t-1]$ at $t \ge 1$, and $(\Delta y)(t) = y(t) = y[t] = y(0)$ at $t \in [0,1)$.

The abstract hybrid functional differential system takes the form

$$\begin{aligned} \mathcal{L}_{11}x + \mathcal{L}_{12}y = \dot{x} - F_{11}x - F_{12}y = f, \\ \mathcal{L}_{21}x + \mathcal{L}_{22}y = \Delta y - F_{21}x - F_{22}y = g. \end{aligned} \tag{9}$$

Here and below $\mathbb{R}^n$ is the space of vectors $\alpha = \mathrm{col}\{\alpha^1, ..., \alpha^n\}$ with real components and the norm $\|\alpha\|_{\mathbb{R}^n}$. Introduce the space L of locally summable $f : [0, \infty) \to \mathbb{R}^n$ with semi-norms $\|f\|_{L[0,T]} = \int_0^T \|f(t)\|_{\mathbb{R}^n} dt$ for all the $T > 0$ and the space D of locally absolutely continuous functions $x : [0, \infty) \to \mathbb{R}^n$ with seminorms

$$\|x\|_{D[0,T]} = \|\dot{x}\|_{L[0,T]} + \|x(0)\|_{\mathbb{R}^n}$$

for all the $T > 0$.

Also introduce the space ℓ of vector functions

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + ... \\ ... + g(N)\chi_{[N,N+1)}(t) + \ldots$$

with the semi-norms $\|g\|_{\ell_T} = \sum_{i=0}^{T} \|g_i\|_{\mathbb{R}^n}$ for all the $T \ge 0$ and the space ℓ_0 of vector functions

$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + \\ + y(1)\chi_{[1,2)}(t) + ... + y(N)\chi_{[N,N+1)}(t) + \ldots$$

with the semi-norms $\|y\|_{\ell_{0T}} = \sum_{i=-1}^{T} \|y_i\|_{\mathbb{R}^n}$ for all the $T \ge -1$.

The operators $\mathcal{L}_{11}, F_{11}: D \to L$, $\mathcal{L}_{12}, F_{12}: \ell_0 \to L$, $\mathcal{L}_{21}, F_{21}: D \to \ell$, $\mathcal{L}_{22}, F_{22}: \ell_0 \to \ell$ are assumed to be continuous linear and Volterra.

Let $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$. Then (9) can be written as $\mathcal{L}\{x, y\} = \mathrm{col}\{f, g\}$.

Suppose that for any $x(0) \in \mathbb{R}^n$ and $y(-1) \in \mathbb{R}^n$ the Cauchy problem for the «model» system $\dot{x} = F_{11}^0 x + F_{12}^0 z + z, \quad \Delta y = F_{21}^0 z + F_{22}^0 y + u,$ where the operators $F_{11}^0: D \to L,\ F_{12}^0: \ell_0 \to L,\ F_{12}^0: \ell_0 \to L,$ $F_{21}^0: D \to \ell,\ F_{22}^0: \ell_0 \to \ell$ are assumed to be continuous linear and Volterra. Then the model system can be written as $\mathcal{L}_0\{x, y\} = \mathrm{col}\{z, u\}$. Suppose its solution can be represented as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ y(-1) \end{pmatrix} + \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}.$$

Here $\mathcal{W}: L \times \ell \to D \times \ell_0$ is the continuous Volterra operator, Cauchy operator for the system, $\mathcal{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $U: \mathbb{R}^n \times \mathbb{R}^n \to D \times \ell_0$ is the fundamental matrix for the system,

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

If the elements $\mathrm{col}\{x, y\}: [0, \infty) \times [-1, \infty) \to \to \mathbb{R}^n \times \mathbb{R}^n$ forming the Banach space $\mathbf{D} \times \mathbf{M}_0 \cong (\mathbf{B} \times \mathbb{R}^n) \times (\mathbf{M} \times \mathbb{R}^n)$ (space $\mathbf{D} \subset D$, space $\mathbf{M}_0 \cong \mathbf{M} \oplus \mathbb{R}^n \subset \ell_0$, space $\mathbf{B} \subset L$, space $\mathbf{M} \subset \ell$, $\mathbf{B}, \mathbf{M}$ are the Banach spaces) have certain specific properties, such as

$$\sup_{t \geq 0} \| x(t) \|_{\mathbb{R}^n} + \sup_{k = -1, 0, 1, \ldots} \| y(k) \|_{\mathbb{R}^n} < \infty,$$

and the Cauchy problem is uniquely solvable for the equation $\mathcal{L}\{x, y\} = \mathrm{col}\{f, g\}$ with the bounded linear operator $\mathcal{L}: \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$, then the solutions of this problem have the same asymptotic properties. This follows from the theorem given below [44].

Theorem 1. Assume $\mathcal{W}: \mathbf{B} \times \mathbf{M} \to \to \mathbf{D} \times \mathbf{M}_0$ is the bounded Cauchy operator of the Cauchy problem for the model equation $\mathcal{L}_0\{x, y\} = \mathrm{col}\{f, g\}$, $\mathrm{col}\{x(0), y(-1)\} = \mathrm{col}\{0, 0\}$ and U is the fundamental matrix of the model equation $\mathcal{L}_0\{x, y\} = \mathrm{col}\{0, 0\}$. Here $\mathcal{L}_0: \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$. Assume the linear operator $\mathcal{L}: \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is bounded, C is the Cauchy operator of the Cauchy problem $\mathcal{L}\{x, y\} = \mathrm{col}\{f, g\}$, $\mathrm{col}\{x(0), y(-1)\} = \mathrm{col}\{0, 0\}$ and X is the fundamental matrix of the equation $\mathcal{L}\{x, y\} = \mathrm{col}\{0, 0\}$. Then for the equality $\mathcal{W}\{\mathbf{B}, \mathbf{M}\} + U\{\mathbb{R}^n, \mathbb{R}^n\} = C\{\mathbf{B}, \mathbf{M}\} + X\{\mathbb{R}^n, \mathbb{R}^n\}$ (10) to hold true it is necessary and sufficient that the operator $\mathcal{L}\mathcal{W}$ (the operator $\mathcal{W}\mathcal{L}$) has a bounded inverse

$$(\mathcal{L}\mathcal{W})^{-1}: \mathbf{B} \times \mathbf{M} \to \mathbf{B} \times \mathbf{M}$$

$$\left((\mathcal{W}\mathcal{L})^{-1}: (\mathbf{D} \times \mathbf{M}_0)^0 \to (\mathbf{D} \times \mathbf{M}_0)^0 \right),$$

where $(\mathbf{D} \times \mathbf{M}_0)^0 = \{\mathrm{col}\{x, y\} \in \mathbf{D} \times \mathbf{M}_0 :$

$$\mathrm{col}\{x(0), y(-1)\} = \mathrm{col}\{0, 0\}\}.$$

Corollary 1 [44]. If the operator $\mathcal{L}: \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is bounded and $\| (\mathcal{L} - \mathcal{L}_0) \mathcal{W} \|_{\mathbf{B} \times \mathbf{M} \to \mathbf{B} \times \mathbf{M}} < 1$ is true or $\| \mathcal{W}(\mathcal{L} - \mathcal{L}_0) \|_{(\mathbf{D} \times \mathbf{M}_0)^0 \to (\mathbf{D} \times \mathbf{M}_0)^0} < 1$ is true, then equality (10) holds true as well.

In the case when (10) holds true (when the solution spaces of the model equation and equation under study coincide), we say that the equation $\mathcal{L}\{x, y\} = \mathrm{col}\{f, g\}$ has the property $\mathbf{D} \times \mathbf{M}_0$, or, in short, the equation is $\mathbf{D} \times \mathbf{M}_0$-stable.

The concept of $\mathbf{D} \times \mathbf{M}_0$-stability relates to the monograph by J.L. Massera and J.J. Shaeffer on the admissibility of pairs of spaces [39] and with the monograph by E.A. Barbashin on the solution property preservation at the accumulation of perturbations [20].

Assume that the model equation [13–19; 28] and Banach space $\mathbf{B}$ with the elements of the space L ($\mathbf{B} \subset L$, this embedding is continuous) are selected so that the solutions of this equation possess asymptotic properties we are interested in.

Suppose, for example, $\sup_{t \geq 0} \| x(t) \|_{\mathbb{R}^n} < \infty$.

Then, putting $\mathcal{L}_{11} x \overset{def}{=} \dot{x} + x = z$, we introduce the Banach space L_∞ of measurable and essentially bounded functions $z: [0, \infty) \to \mathbb{R}^n$ with the norm $\operatorname{vrai\,sup}_{t \geq 0} \| z(t) \|_{\mathbb{R}^n} < \infty$ as the Banach space $\mathbf{B}$. The space $D(\mathcal{L}_{11}, L_\infty)$ generated by the model equation consists of solutions of the form

$$x(t) = (\mathcal{W}_{11} z)(t) + (\mathcal{U}_{11} \alpha)(t) = \int_0^t e^{-(t-s)} z(s) ds + \alpha e^{-t}$$

$$(\alpha \in \mathbb{R}^n,\ z \in L_\infty).$$

These solutions are bounded ($\sup_{t \geq 0} \| x(t) \|_{\mathbb{R}^n} < \infty$) and their derivative $\dot{x} = -x + z$ is in L_∞. All the solutions of this equation form the Banach space with the norm

$$\| x \|_{D(\mathcal{L}_{11}, L_\infty)} = \\ = \operatorname{vrai\,sup}_{t \geq 0} \| \dot{x}(t) + x(t) \|_{\mathbb{R}^n} + \| x(0) \|_{\mathbb{R}^n} < \infty,$$

which is linearly isomorphic to the Sobolev space $W_\infty^{(1)}[0, \infty)$ with the norm

$$\| x \|_{W_\infty^{(1)}[0, \infty)} = \sup_{t \geq 0} \| x(t) \|_{\mathbb{R}^n} + \operatorname{vrai\,sup}_{t \geq 0} \| \dot{x}(t) \|_{\mathbb{R}^n}.$$

Here in after this space is referred to as W_{L_∞}. Here $W_{L_\infty} \subset D$, this embedding is continuous.

Similarly, for the Banach space $\mathbf{B} \subset L$ we introduce the Banach space $D(\mathcal{L}_{11}, \mathbf{B})$ with the norm

$$\| x \|_{D(\mathcal{L}_{11}, \mathbf{B})} = \| \dot{x} + x \|_{\mathbf{B}} + \| x(0) \|_{\mathbb{R}^n}.$$

Here the embedding $\mathbf{B} \subset L$ is assumed to be continuous. Assume that the operator $\mathcal{W}_{11}$ acts continuously from the space $\mathbf{B}$ into the space $\mathbf{B}$, and the operator $\mathcal{U}_{11}$ acts from space $\mathbb{R}^n$ into the space $\mathbf{B}$. This condition is equivalent to the fact [13–16] that the space $D(\mathcal{L}_{11},\mathbf{B})$ is linearly isomorphic to the Sobolev space with the norm

$$\| x \|_{W^{(1)}_{\mathbf{B}}[0,\infty)} = \| \dot{x} \|_{\mathbf{B}} + \| x \|_{\mathbf{B}} .$$

Hereinafter this space is referred to as $\boldsymbol{W}_{\mathbf{B}}$ ($\boldsymbol{W}_{\mathbf{B}} \subset D$, this embedding is continuous).

The equation $\mathcal{L}_{11}x = z$ with the operator $\mathcal{L}_{11} : \boldsymbol{W}_{\mathbf{B}} \to \mathbf{B}$ is $D(\mathcal{L}_{11},\mathbf{B})$-stable if and only if it is strongly $\mathbf{B}$-stable. $\mathcal{L}_{11}x = z$ is strongly $\mathbf{B}$- stable if for any $z \in \mathbf{B}$ each solution x of this equation has the property $x \in \mathbf{B}$ and $\dot{x} \in \mathbf{B}$ [14, Ch. IV, § 4.6; 4].

2.2. Let us consider the scheme from subsection 2.1 for two equations (9). The operators $\mathcal{L}_{11} : D \to L,\ \ \mathcal{L}_{12} : \ell_0 \to L,\ \ \mathcal{L}_{21} : D \to \ell,\ \ \mathcal{L}_{22} : \ell_0 \to \ell$ are considered as reduction to pairs $(\boldsymbol{W}_{\mathbf{B}},\mathbf{B}),\ (\mathbf{M}_0,\mathbf{B}),\ (\boldsymbol{W}_{\mathbf{B}},\mathbf{M}),\ (\mathbf{M}_0,\mathbf{M})$. These operators are assumed to be Volterra linear and bounded operators.

Assume that the general solution of the equation $\mathcal{L}_{22}y = g$ for $g \in \mathbf{M}$ is the space of $\mathbf{M}_0$ and is represented by the Cauchy formula:

$$y[t] = Y_{22}[t]y(-1) + \sum_{s=0}^{[t]} C_{22}[t,s]g[s].$$

Let

$$(C_{22}g)[t] = \sum_{s=0}^{[t]} C_{22}[t,s]g[s],$$

$$(Y_{22}y(-1))[t] = Y_{22}[t]y(-1).$$

Then every solution y of the second equation in (9) has the form:

$$y = -C_{22}\mathcal{L}_{21}x + Y_{22}y(-1) + C_{22}g.$$

Substituting the first equation into (9) we obtain

$$\mathcal{L}_{11}x + \mathcal{L}_{12}y = \mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x + \mathcal{L}_{12}Y_{22}y(-1) + \mathcal{L}_{12}C_{22}g = f,$$

$$\mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x = f_1 = f - \mathcal{L}_{12}Y_{22}y(-1) - \mathcal{L}_{12}C_{22}g.$$

Let $\mathcal{L} = \mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}$, then the first equation in (9) takes the form of $\mathcal{L}x = f_1$.

Suppose the Volterra operator $\mathcal{L} : (\boldsymbol{W}_{\mathbf{B}})^0 \to \mathbf{B}$ is Volterra invertible (the Cauchy problem for $\mathcal{L}x = f_1$ possess the following property: at any $f_1 \in \mathbf{B}$ its solutions are $x \in \boldsymbol{W}_{\mathbf{B}}$). Thus, we solved the problem, when for equation (9) at any $\{f, g\} \in \mathbf{B} \times \mathbf{M}$ its solutions are $\{x, y\} \in \boldsymbol{W}_{\mathbf{B}} \times \mathbf{M}$.

Example 1. Consider the following two equations:

$$\dot{x}(t) + ax(t) + by[t] = f(t),\ t \in [0,\infty), \qquad (11)$$
$$y[t] - dy[t-1] + cx[t] = g[t],\ \ t \in [0,\infty),$$

putting

$$y(0) - dy(-1) + c(0) = y[t] - dy[t-1] + cx[t] =$$
$$= g[t] = g(0),\ \ t \in [0,1).$$

Let us introduce the following spaces:

$$\ell_{\infty 0} = \{ y \in \ell_0 : \| y \|_{\ell_{\infty 0}} = \sup_{k=-1,0,1,\cdots} \| y(k) \|_{\mathbb{R}^n} < +\infty \},$$

$$\ell_{\infty} = \{ g \in \ell : \| g \|_{\ell_\infty} = \sup_{k=0,1,\cdots} \| g(k) \|_{\mathbb{R}^n} < +\infty \}.$$

If we introduce the operator $(Sy)(t) = dy(t-1),\ \ t \ge 1,\ \ \ (Sy)(t) = 0,\ t \in [0,1)$, then the second equation takes the form

$$y(t) - (Sy)(t) + cx(t) =$$
$$= g_1(t) = g(t) + dy(t-1),\ t \in [0,1),$$
$$y(t) - (Sy)(t) + cx(t) = g(t),\ t \in [1,\infty).$$

Let us consider the operator $S : \ell_\infty \to \ell_\infty$. We know that the operator $(I - S) : \ell_\infty \to \ell_\infty$ is Volterra invertible if and only if the spectral radius $\rho_{\ell_\infty}(S)$ is less than one. For S the condition $\rho_{\ell_\infty}(S) < 1$ is equivalent to the inequality $|d| < 1$ [47, p. 87, p. 140].

Let us put

$$(\mathcal{L}_{11}x)(t) = \dot{x}(t) + ax(t),\ t \ge 0,$$
$$(\mathcal{L}_{12}y)[t] = by[t],\ \ t \ge 0,$$
$$(\mathcal{L}_{21}x)(t) = cx[t],\ \ t \ge 0,$$
$$(\mathcal{L}_{22}x)[t] = y[t] - (Sx)[t],\ \ t \ge 0 .$$

Now we build the Cauchy function C_{22} and the fundamental solution Y_{22} for the equation $y[t] - dy[t-1] = g[t]$:

$$y[t] = d^{t+1}y(-1) + \sum_{s=0}^{[t]} g(s)d^{[t]-s} =$$
$$= Y_{22}[t]y(-1) + (C_{22}g)[t].$$

Out of this we can express $y[t]$ of the second equation of (11):

$$y[t] = d^{t+1}y(-1) + \sum_{s=0}^{[t]} (g[s] - cx[s])d^{[t]-s} =$$
$$= Y_{22}[t]y(-1) + (C_{22}(g - cx))[t].$$

Substituting the obtained y into the first formula of (9) (or (11)) we get

$$(\mathcal{L}_{11}x)(t) + (\mathcal{L}_{12}y)[t] = \dot{x}(t) + ax(t) +$$
$$+ bd^{t+1}y(-1) + b\sum_{s=0}^{[t]} (g[s] - cx[s])d^{[t]-s} = f(t).$$

Further we have

$$(\mathcal{L}x)(t) = ((\mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21})x)(t) = \dot{x}(t) + ax(t) -$$
$$-bc\sum_{s=0}^{[t]} x[s]d^{[t]-s} = f_1(t) = f(t) - bd^{t+1}y(-1) -$$
$$-b\sum_{s=0}^{[t]} g[s]d^{[t]-s}.$$

It is evident that $f_1 \in L_\infty$ if $|d| < 1$.

Let us write the Cauchy formula for

$$(\mathcal{L}_{11}x)(t) = bc\sum_{s=0}^{[t]} x[s]d^{[t]-s} + f_1(t):$$

$$x(t) = X_{11}(t)x(0) + \int_0^t C_{11}(t,s)(bc\sum_{i=0}^{[s]} x[i]d^{[s]-i} +$$
$$+ f_1(s))ds.$$

We have $X_{11}(t)=e^{-at}$, $C_{11}(t,s)=e^{-a(t-s)}$. For a positive $a>0$, we can estimate:

$$\sup_{t\ge 0}\left|\int_0^t C_{11}(t,s)bc\sum_{i=0}^{[s]}x[i]d^{[s]-i}ds\right|\le$$

$$\le \sup_{t\ge 0}\left(|bc|\,e^{-at}\int_0^t e^{as}\sum_{i=0}^{[s]}|d|^{[s]-i}ds\cdot\|x\|_{L_\infty}\right)\le$$

$$\le |bc|\cdot\frac{1}{1-|d|}\cdot\sup_{t\ge 0}e^{-at}\int_0^t e^{as}ds\cdot\|x\|_{L_\infty}\le$$

$$\le\frac{1}{a}\cdot|bc|\cdot\frac{1}{1-|d|}\cdot\|x\|_{L_\infty}.$$

Hence, the norm of operator $bC_{11}cC_{22}$ is less than 1 if

$$|bc|<a(1-|d|).$$

Thus, for any $f_1\in L_\infty$ the solution x to the problem $\mathcal{L}x=f_1$ lies within the space L_∞, and, besides, the derivative of the solution $\dot{x}$ is in the space L_∞. This establishes that for any $f_1\in L_\infty$ the solution x of the problem $\mathcal{L}x=f_1$ is in the space W_{L_∞}.

Thus, we solved the problem when at any $\{f,g\}\in L_\infty\times\ell_\infty$ for equation (11) its solutions are $\{x,y\}\in\mathbf{D}\cong W_{L_\infty}\times\ell_{\infty 0}$.

2.3. Let us use the ability of the hybrid system to be reduced to a linear difference equation defined on a discrete set of points.

For equation (9) we use the designations given in subsections 2.1 and 2.2.

Assume that the general solution of the equation $\mathcal{L}_{11}x=f$ for $f\in L$ is a member of the space D and is represented by the Cauchy formula:

$$x(t)=X_{11}(t)x(0)+\int_0^t C_{11}(t,s)f(s)ds.$$

Since $(C_{11}f)(t)=\int_0^t C_{11}(t,s)f(s)ds$ and $(X_{11}x(0))(t)=X_{11}(t)x(0)$, we have, for $x\in D$, the representation $x=X_{11}x(0)+C_{11}f$.

The first variable x can be estimated out of the first equation in (8):

$$x=-C_{11}\mathcal{L}_{12}y+X_{11}x(0)+C_{11}f.$$

By the use of this substitution in the second equation of (9) we obtain:

$$\mathcal{L}_{21}x+\mathcal{L}_{22}y\ =$$

$$=\ -\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y+\mathcal{L}_{21}X_{11}x(0)+\mathcal{L}_{21}C_{11}f+\ \mathcal{L}_{22}y=g,$$

$$-\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y+\mathcal{L}_{22}y=g_1=g-\mathcal{L}_{21}X_{11}x(0)-\mathcal{L}_{21}C_{11}f.$$

Put $\mathcal{L}=\mathcal{L}_{22}-\mathcal{L}_{21}C_{11}\mathcal{L}_{12}$, then the second equation in (9) takes the form $\mathcal{L}y=g_1$.

Suppose that the Volterra operator $\mathcal{L}:(\mathbf{M}_0)^0\to\mathbf{M}$ is Volterra invertible (for the Cauchy problem for $\mathcal{L}y=g_1$ at any $g_1\in\mathbf{M}$ its solutions are $x\in\mathbf{M}_0$). Thus, we solved the problem, in the case that at any $\{f,g\}\in\mathbf{B}\times\mathbf{M}$ solutions of (9) are $\{x,y\}\in\mathbf{D}\times\mathbf{M}_0$.

Example 2. Let us consider two equations:

$$\dot{x}(t)+ax(t)+by[t]=f(t),\ \ t\in[0,\infty),\qquad(12)$$
$$y[t]-dy[t-1]+cx[t]=g[t],\ \ t\in[0,\infty).$$

Using the Cauchy formula for x, the first equation in (12) can be written as

$$x(t)=X_{11}(t)x(0)+\int_0^t C_{11}(t,s)(f(s)-by[s])ds$$

or

$$x(t)=e^{-at}x(0)+\int_0^t e^{-a(t-s)}(f(s)-by[s])ds.$$

Substituting x into the second equation in (12) we obtain

$$y[t]-dy[t-1]+c(e^{-a[t]}x(0)+$$
$$+\int_0^{[t]}e^{-a([t]-s)}(f(s)-by[s])ds)=g[t],$$

$$y[t]-dy[t-1]-bc\int_0^{[t]}e^{-a([t]-s)}y[s]ds=g_1[t]=$$

$$=g[t]-ce^{-a[t]}x(0)-c\int_0^{[t]}e^{-a([t]-s)}f(s)ds.$$

Calculating the integral

$$bc\int_0^{[t]}e^{-a([t]-s)}y[s]ds=bce^{-a[t]}\int_0^{[t]}e^{as}y[s]ds=$$

$$=bce^{-a[t]}\sum_{i=0}^{[t]-1}y[i]\int_i^{i+1}e^{as}ds=bce^{-a[t]}\sum_{i=0}^{[t]-1}y[i]\int_i^{i+1}e^{as}ds=$$

$$=bce^{-a[t]}\sum_{i=0}^{[t]-1}y[i](e^{a(i+1)}-e^{ai})/a=$$

$$=\frac{bc}{a}\sum_{i=0}^{[t]-1}y[i](e^{-a([t]-i-1)}-e^{-a([t]-i)}),$$

we obtain the equation

$$y[t]-dy[t-1]-\frac{bc}{a}\sum_{i=0}^{[t]-1}y[i](e^{-a([t]-i-1)}-e^{-a([t]-i)})=$$
$$=g_1[t],\ \ t\in[0,\infty).$$

Define the operator K by the equality

$$(Ky)[t]=\frac{bc}{a}\sum_{i=0}^{[t]-1}y[i](e^{-a([t]-i-1)}-e^{-a([t]-i)}).$$

Assuming $a>0$, let us estimate the norm $\|K\|_{\ell_{\infty 0}\to\ell_\infty}$:

$$\|Ky\|_{\ell_\infty}=\sup_{k=0,1,2,\ldots}\left|\frac{bc}{a}\sum_{i=0}^{[t]-1}y[i](e^{-a([t]-i-1)}-e^{-a([t]-i)})\right|\le$$

$$\le\|y\|_{\ell_{\infty 0}}\cdot\frac{|bc|}{a}\sup_{k=0,1,2,\ldots}(1-e^{-ak})=\|y\|_{\ell_{\infty 0}}\cdot\frac{|bc|}{a}.$$

Next, we estimate the norm

$$\|(I-S)^{-1}K\|_{\ell_\infty\to\ell_\infty}:$$

$$\|(I-S)^{-1}K\|_{\ell_\infty\to\ell_\infty}\le\|(I-S)^{-1}\|_{\ell_\infty\to\ell_{\infty 0}}\cdot\|K\|_{\ell_{\infty 0}\to\ell_\infty}\le$$

$$\le\frac{1}{1-|d|}\cdot\frac{|bc|}{a}.$$

Thus we find that $\|(I-S)^{-1}K\|_{\ell_\infty\to\ell_\infty}$ is less than 1 if

$$|bc|<a(1-|d|).$$

So, for any $g_1 \in \ell_\infty$ the solution y of the equation $\mathcal{L}y = g_1$ lies within ℓ_∞.

Thus, we solved the problem, when for (12) at any $\{f,g\} \in L_\infty \times \ell_\infty$ its solution are $\{x,y\} \in W_{L_\infty} \times \ell_{\infty 0}$.

2.4. Here we use the ability of the original hybrid system to be reduced to the auxiliary linear integral equation on the base of the *W*-method.

Let us apply Corollary 1 from 2.1.

Example 3. Consider the two equations:

$$\dot{x}(t) + ax(t) + by[t] = f(t),\ t \in [0,\infty),$$
$$y[t] - dy[t-1] + cx[t] = g[t],\ t \in [0,\infty). \tag{13}$$

Using the Cauchy formula for x, the first equation in (13) can be rewritten in the form

$$x(t) = X_{11}(t)x(0) + \int_0^t C_{11}(t,s)(f(s) - by[s])ds,$$
$$x = X_{11}x(0) + C_{11}(f - by).$$

Let us construct the Cauchy function C_{22} and the fundamental solution Y_{22} for the equation $y[t] - dy[t-1] = g[t]$:

$$y[t] = d^{t+1}y(-1) + \sum_{s=0}^{[t]} g(s)d^{[t]-s} =$$
$$= Y_{22}[t]y(-1) + (C_{22}g)[t].$$

From this we can the express $y[t]$ from the second equation in (13):

$$y[t] = d^{t+1}y(-1) + \sum_{s=0}^{[t]} (g[s] - cx[t])d^{[t]-s},$$
$$y = Y_{22}y(-1) + C_{22}(g - cx).$$

Let us consider the model equation in the form of a system

$$\dot{x}(t) + ax(t) = f(t),\quad t \in [0,\infty),$$
$$y[t] - dy[t-1] = g[t],\quad t \in [0,\infty).$$

It is known that when $a > 0$ and $|d| < 1$, this system has the following property: at any $f \in L_\infty$, $g \in \ell_\infty$ it follows that $x \in W_{L_\infty}$, $y \in \ell_{\infty 0}$.

We check when this property is fulfilled for system (13). For that it is sufficient to verify the assertion of Corollary 1 from subsection 2.2: if

$$\|(\mathcal{L} - \mathcal{L}_0)\mathcal{W}\|_{L_\infty \times \ell_\infty \to L_\infty \times \ell_\infty} < 1$$

(or

$$\|\mathcal{W}(\mathcal{L} - \mathcal{L}_0)\|_{(W_{L_\infty} \times \ell_{\infty 0})^0 \to (W_{L_\infty} \times \ell_{\infty 0})^0} < 1)$$

holds true, then the operator $\mathcal{LW}$ (operator $\mathcal{WL}$) has a bounded inverse

$$(\mathcal{LW})^{-1} : L_\infty \times \ell_\infty \to L_\infty \times \ell_\infty$$
$$((\mathcal{WL})^{-1} : (W_{L_\infty} \times \ell_{\infty 0})^0 \to (W_{L_\infty} \times \ell_{\infty 0})^0).$$

Here we have

$$\mathcal{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix},$$
$$(\mathcal{L} - \mathcal{L}_0)\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} 0 & by[t] \\ cx[t] & 0 \end{pmatrix}.$$

Variant I. Consider the case where the second condition takes place.

By Lemma 2.4.2 from [14] (Lemma 2 from [13]) the $C \times \ell_{\infty 0}$-stability of (13) can be studied instead of the $W_{L_\infty} \times \ell_{\infty 0}$-stability of this system. Here $C = C[0,\infty)$ is the Banach space of bounded functions $x : [0,\infty) \to \mathbb{R}^n$ with the norm $\|x\|_C = \sup_{t \ge 0} \|x(t)\|_{\mathbb{R}^n}$.

Multiplying

$$\begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}\begin{pmatrix} 0 & by[t] \\ cx[t] & 0 \end{pmatrix} = \begin{pmatrix} 0 & (C_{11}by)(t) \\ (C_{22}cx)[t] & 0 \end{pmatrix},$$

we calculate

$$(C_{11}by)(t) = b\int_0^t e^{-a(t-s)}y[s]ds,$$
$$(C_{22}cx)[t] = c\sum_{s=0}^{[t]} x[t]d^{[t]-s}.$$

Now let us estimate the operator norm $\|C_{11}b\|_{\ell_{\infty 0} \to L_\infty}$:

$$|b|\sup_{t\ge 0}|(C_{11}y)(t)| = |b|\sup_{t \ge 0}\left|\int_0^t e^{-a(t-s)}y[s]ds\right| =$$
$$= |b|\sup_{t\ge 0}\left|\int_0^{[t]+\{t\}} e^{-a([t]+\{t\}-s)}y[s]ds\right| =$$
$$= |b|\sup_{t\ge 0}\Big(\int_{\{t\}}^{[t]} e^{-a([t]+\{t\}-s)}|y[s]|ds +$$
$$+ \int_0^{\{t\}} e^{-a([t]+\{t\}-s)}|y[s]|ds\Big) \le$$
$$\le |b|\sup_{t\ge 0}\int_0^{[t]} e^{-a([t]+\{t\}-s)}|y[s]|ds +$$
$$+ |b|\sup_{t\ge 0}\int_0^{\{t\}} e^{-a([t]+\{t\}-s)}|y[s]|ds \le$$
$$\le |b|\sup_{t\ge 0} e^{-a\{t\}}\int_0^{[t]} e^{-a([t]-s)}|y[s]|ds +$$
$$+ |b|\cdot|y(0)|\sup_{t\ge 0} e^{-at}\int_0^{\{t\}} e^{as}ds \le$$
$$\le |b|\sup_{t\ge 0} e^{-a\{t\}-a[t]}\sum_{i=0}^{[t]-1}|y[i]|\int_i^{i+1} e^{as}ds + |b|\cdot|y(0)|\frac{e^a - 1}{a} \le$$
$$\le \frac{|b|}{a}\sup_{t\ge 0} e^{-a\{t\}-a[t]}\sum_{i=0}^{[t]-1}|y[i]|(e^{a(i+1)} - e^{ai}) + |b|\cdot|y(0)|\frac{e^a - 1}{a} \le$$
$$\le \frac{|b|}{a}\sup_{t\ge 0} e^{-a\{t\}-a[t]}\sum_{i=0}^{[t]-1}(e^{a(i+1)} - e^{ai})ds \cdot \|y\|_{\ell_\infty} +$$
$$+ |b|\cdot|y(0)|\frac{e^a - 1}{a} \le$$
$$\le \frac{|b|}{a}\sup_{t\ge 0} e^{-a\{t\}} \cdot \sup_{k=0,1,2,\ldots}\sum_{i=0}^{k-1}(e^{-a(k-i-1)} - e^{-a(k-i)}) \cdot \|y\|_{\ell_\infty} +$$
$$+ |b|\cdot|y(0)|\frac{e^a - 1}{a} \le$$
$$\le \frac{|b|}{a}\sup_{t\ge 0} e^{-a\{t\}} \cdot \sup_{k=0,1,2,\ldots}(1 - e^{-ak}) \cdot \|y\|_{\ell_\infty} +$$

$$+|b|\cdot|y(0)|\frac{e^a-1}{a}\le\frac{|b|}{a}\max\{1,e^a-1\}\|y\|_{\ell_{\infty 0}}.$$

That is, when $0<a\le\ln 2$, then $\|C_{11}b\|_{\ell_{\infty 0}\to L_\infty}<\frac{|b|}{a}$.

Estimate the operator norm $\|C_{22}c\|_{C\to L_\infty}$:

$$\sup_{t\ge 0}|c\sum_{i=0}^{[t]}x[i]d^{[t]-i}|\le|c|\cdot\sup_{t\ge 0}\sum_{i=0}^{[s]}|d|^{[s]-i}\cdot\|x\|_C\le$$

$$\le|c|\cdot\frac{1}{1-|d|}\cdot\|x\|_C,$$

$$\|C_{22}c\|_{C\to L_\infty}\le|c|\cdot\frac{1}{1-|d|}.$$

Let us consider the cases of the following norms in the space $\mathbb{R}^2$:

$$\|(x,y)\|_1=|x|+|y|,$$

$$\|(x,y)\|_2=(x^2+y^2)^{1/2},$$

$$\|(x,y)\|_\infty=\max\{|x|,|y|\}.$$

Then, for the corresponding norms of the matrix $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, we have

$$\|A\|_1=\sup\{|a_{11}|+|a_{12}|,|a_{21}|+|a_{22}|\},$$

$$\|A\|_2\le(a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)^{1/2},$$

$$\|A\|_\infty=\max\{|a_{11}|+|a_{21}|,|a_{12}|+|a_{22}|\}.$$

Suppose $0<a\le\ln 2$, then, for above cases, we obtain

$$\|\mathcal{W}(\mathcal{L}-\mathcal{L}_0)\|_{(C\times\ell_{\infty 0})^0\to(C\times\ell_{\infty 0})^0}\le\frac{|b|}{a}+\frac{|c|}{1-|d|},$$

$$\|\mathcal{W}(\mathcal{L}-\mathcal{L}_0)\|_{(C\times\ell_{\infty 0})^0\to(C\times\ell_{\infty 0})^0}\le\left(\frac{b^2}{a^2}+\frac{c^2}{(1-|d|)^2}\right)^{1/2},$$

$$\|\mathcal{W}(\mathcal{L}-\mathcal{L}_0)\|_{(C\times\ell_{\infty 0})^0\to(C\times\ell_{\infty 0})^0}\le\max\left\{\frac{|b|}{a},\frac{|c|}{1-|d|}\right\},$$

respectively.

So, for $0<a\le\ln 2$, we obtain the conditions: either $\frac{|b|}{a}+\frac{|c|}{1-|d|}<1$, or $\frac{b^2}{a^2}+\frac{c^2}{(1-|d|)^2}<1$, or $\max\left\{\frac{|b|}{a},\frac{|c|}{1-|d|}\right\}<1$.

Thus, we solved the problem, when for equation (13) at any $\{f,g\}\in L_\infty\times\ell_\infty$ its solutions are $\{x,y\}\in C\times\ell_{\infty 0}$, or $\{x,y\}\in W_{L_\infty}\times\ell_{\infty 0}$.

Variant II. Consider $\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{L_\infty\times\ell_\infty\to L_\infty\times\ell_\infty}$. Let us study when $\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{L_\infty\times\ell_\infty\to L_\infty\times\ell_\infty}<1$.

Let us put: $(\bar{I}y)(t)=y[t]$ and $(\bar{I}x)(t)=x[t]$.

Multiply

$$\begin{pmatrix}0&b\bar{I}\\c\bar{I}&0\end{pmatrix}\begin{pmatrix}C_{11}&0\\0&C_{22}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}(t)=\begin{pmatrix}b(C_{22}y)[t]\\c(C_{11}x)[t]\end{pmatrix}.$$

Then calculate

$$c(C_{11}x)[t]=c\int_0^{[t]}e^{-a([t]-s)}x(s)ds,$$

$$b(C_{22}y)[t]=b\sum_{s=0}^{[t]}y[t]d^{[t]-s}.$$

Estimate the operator norm $\|cC_{11}\|_{L_\infty\to\ell_\infty}$. Suppose $a>0$ and calculate the integral

$$|c\int_0^{[t]}e^{-a([t]-s)}x(s)ds|=|ce^{-a[t]}\int_0^{[t]}e^{as}x(s)ds|=$$

$$=|ce^{-a[t]}\sum_{i=0}^{[t]-1}\int_i^{i+1}e^{as}x(s)ds|\le$$

$$\le|c|e^{-a[t]}\sum_{i=0}^{[t]-1}\|x\|_{L_\infty[i,i+1]}\int_i^{i+1}e^{as}ds=$$

$$=|c|e^{-a[t]}\sum_{i=0}^{[t]-1}\|x\|_{L_\infty[i,i+1]}(e^{a(i+1)}-e^{ai})/a=$$

$$=\frac{|c|}{a}\sum_{i=0}^{[t]-1}\|x\|_{L_\infty[i,i+1]}(e^{-a([t]-i-1)}-e^{-a([t]-i)}).$$

Define the operator K by the equality

$$(Kx)[t]=\frac{|c|}{a}\sum_{i=0}^{[t]-1}\|x\|_{L_\infty[i,i+1]}(e^{-a([t]-i-1)}-e^{-a([t]-i)}).$$

Estimate the norm $\|K\|_{L_\infty\to\ell_\infty}$:

$$\|Kx\|_{\ell_\infty}=\sup_{[t]=1,2,\ldots}|\frac{c}{a}\sum_{i=0}^{[t]-1}\|x\|_{L_\infty[i,i+1]}(e^{-a([t]-i-1)}-e^{-a([t]-i)})|\le$$

$$\le\|y\|_{\ell_{\infty 0}}\cdot\frac{|bc|}{a}\sup_{k=0,1,2,\ldots}(1-e^{-ak})=\|x\|_{L_\infty}\cdot\frac{|c|}{a}.$$

Then estimate the norm $\|bC_{22}\|_{\ell_\infty\to L_\infty}$:

$$\sup_{t\ge 0}|b\sum_{i=0}^{[t]}y[i]d^{[t]-i}|\le|b|\cdot\sup_{t\ge 0}\sum_{i=0}^{[t]}|d|^{[t]-i}\cdot\|y\|_{\ell_\infty}\le$$

$$\le|b|\cdot\frac{1}{1-|d|}\cdot\|y\|_{\ell_\infty},$$

$$\|bC_{22}\|_{\ell_\infty\to L_\infty}\le|b|\cdot\frac{1}{1-|d|}.$$

Let us consider separately three cases of determination of the norm in the space $\mathbb{R}^2$: $\|(\cdot,\cdot)\|_1$, $\|(\cdot,\cdot)\|_2$, $\|(\cdot,\cdot)\|_\infty$.

Suppose $a>0$, then in the case of the first norm we obtain

$$\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{L_\infty\times\ell_\infty\to L_\infty\times\ell_\infty}\le\frac{|c|}{a}+\frac{|b|}{1-|d|},$$

in the case of the second norm we obtain

$$\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{L_\infty\times\ell_\infty\to L_\infty\times\ell_\infty}\le\left(\frac{c^2}{a^2}+\frac{b^2}{(1-|d|)^2}\right)^{1/2},$$

in the case of the third norm we obtain

$$\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{L_\infty\times\ell_\infty\to L_\infty\times\ell_\infty}\le\max\left\{\frac{|c|}{a},\frac{|b|}{1-|d|}\right\}.$$

So we obtain the following three conditions of stability: $\frac{|c|}{a}+\frac{|b|}{1-|d|}<1$, $\frac{c^2}{a^2}+\frac{b^2}{(1-|d|)^2}<1$, $\max\left\{\frac{|c|}{a},\frac{|b|}{1-|d|}\right\}<1$.

Thus, we solved the problem when for equation (13) at any $\{f, g\} \in L_\infty \times \ell_\infty$ its solution $\{x, y\}$ are elements of the space $W_{L_\infty} \times \ell_{\infty 0}$.

The background of the solutions stability of linear difference equations and LFDEA is presented in [45], where also a description of application of linear HFDSA for modeling investment development of high-tech industries is given.

3. Software for modeling and analysis

In this Section we restrict ourselves to some very recent results in modeling and analysis as applied to five topical problems: Internal rating based modeling, Analysis of shocks and their trigger mechanisms, Developing a typology of financial market participants, Analysis of financial market regulation consequences, and Simulation of financial markets.

3.1. Internal rating based modeling

Building an internal rating based model for a company helps tackle a number of practical aspects of building IRB models that involve the definition of discretization parameters and dynamic transformations of factors using macroeconomic variables as factors and mapping the model to an international scale.

An important stage in the IRB approach implementation is the development of a qualitative model to assess the probability of default of corporate counterparties that allows evaluating their credit quality efficiently. Developing such a model is a nontrivial task that might involve numerous technical details and complexities. The algorithm for creating the probability of default (PD) model has the following steps:

1. Identification of a set of potential factors of the model such as groups of financial ratios, macro-indicators; calculation of the selected financial ratios using data from financial statements;
2. Analysis of the financial ratios (tracing of ROC curves, calculation of Area Under Curve (AUC) ratios, selection of measures that have maximum predictive power, analysis of outliers, and discretization);
3. Testing of all possible variants of a logistic regression model that are evaluated only with use of financial ratios; selection of the best model variant;
4. Adding of micro-factors to the model defined during step 3; selection of the best model variant;
5. Appraisal of the stability of the model ratios in different periods.

In the paper [25], 18 financial ratios are considered to create the PD model. These ratios can be divided into the following groups:

– Debt to equity ratios;
– Profitability ratios;
– Liquidity ratios;
– Turnover ratios;
– Returns-to-scale ratios.

The sample used for the analysis and PD model creation contains data from annual statements (balance sheets, income statements) of more than 8,000 Russian entities from the non-financial sector for the period of seven years. In total, it includes about 50,000 observations, where default observations account for 2.3%. The original sample is divided into a training set and a validation set using a 70%/30% ratio, where the default levels must be equal in each set.

The authors built a series of models that achieve prediction accuracies in the range of 80–90% by AUC criterion [26]. The obtained results are successfully applied for different customers in the real and banking sectors.

Based on the empiric study findings the authors arrived to the following practical conclusions:

1. The predictive power of the factors can significantly decrease after dynamic transformations (increments, growth rates).
2. Discretization of factor values allows improving their predictive power and go to a monotonous ROC curve.
3. It makes sense to include macroeconomic factors in the model when the sample is representative in relation to the macroeconomic cycle.

Modeling was done using the PROGNOZ. Credit Risk software solution that provides comprehensive BI support to analyze financial position of counterparties and rerun models using different measures and different counterparty groups (Fig. 1).

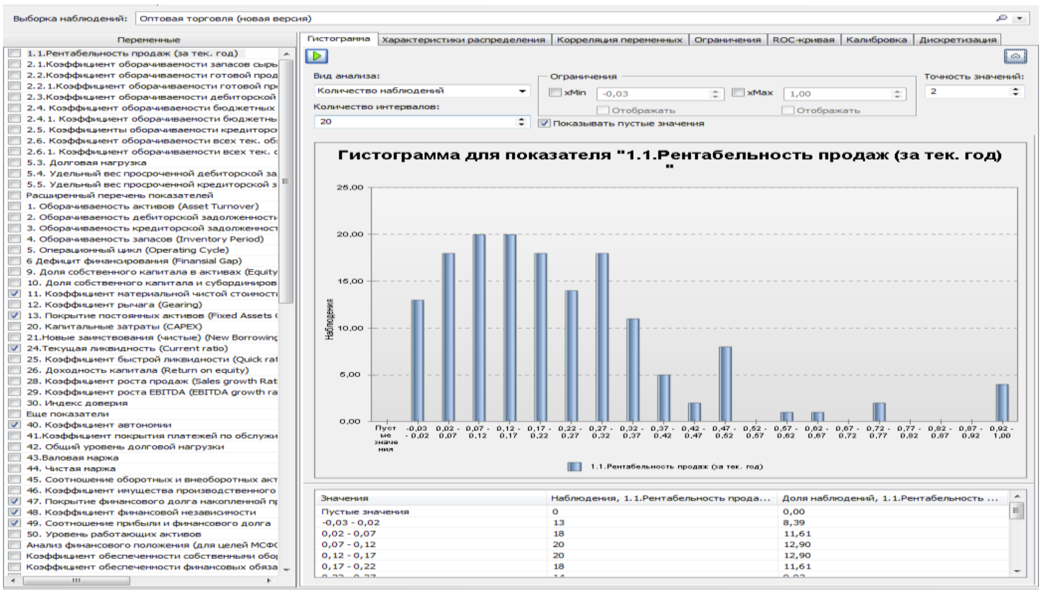

Fig. 1. IRB modeling using the software solution

3.2. Analysis of shocks and their trigger mechanisms

Gaining insight into a price formation mechanism is one of most relevant problems in the modern economic theory. There are many papers on stylized facts of price series, but the reason why these facts exist is not clear so far. Studying the dynamics of market characteristics in proximity to price shocks can logically provide valuable insights into the nature of this phenomenon. Based on a statistical approach, we have tried to answer the question: What happens in close proximity (when considering high-frequency data) to a leap in prices? An attempt to study shocks along with preshock and postshock market behavior is not new. The concept of a market shock is relative and should be considered in the context of a timeline and level of local volatility. In paper [25], three event types are considered, each of which is defined by an appropriate timeline (hours, minutes, and ticks). Later they are denoted as macro-, meso-, and microevents respectively and analyzed using four key market metrics such as price level, trade imbalance, trading volume, and bid-ask spread.

In paper [24], three timelines are determined:

– Level of hours (the macrolevel);
– Level of minutes (the mesolevel);
– Level of ticks (the microlevel).

To identify events at the macro- and mesolevels, a series of minute-level prices is generated that is calculated as a half-sum of the best bid price and best ask price (or mid-point price) at the end of each minute. At the microlevel, tick-level price changes are used. A tick means any change in price caused by the execution of orders.

To identify shocks at the macrolevel, two filters (absolute and relative filters) are combined and price changes are considered as shocks when both filters detect such changes simultaneously.

To identify an extreme event at the mesolevel, a filter is used, where an absolute value of one-minute returns is compared against moving average of one-minute returns. A shock is defined as a time point when the absolute value of one-minute returns is s times greater than the moving average of one-minute returns.

For the tick timeline, we used the Nanex methodology, where a downward (upward) price movement is defined as a shock, if the price had to tick down (up) at least 10 times before ticking up (down) – all within 2 seconds and the price change had to exceed 0.8%. A tick means a price change caused by a trade(s). To apply this type of filter, we generated a series of tick-level prices based on trade data.

At the macrolevel, we identified 1,820 events for the analysis period of four months. At the mesolevel, we identified 13,368 events or 461 per each stock in average or 5.5 events per day. Similarly to the macrolevel timeline, the frequency of identified events varies greatly among stocks: from 0.4 to 17 shocks per day. We have revealed an inverse relationship between the number of identified shocks at the mesolevel and the average number of trades/bids [25, Fig. 1]. The greater the number of trades/bids per day on average, i.e. the higher is a stock liquidity, less shocks it has at the mesolevel. At the microlevel, we identified 369 events, on average 3.3 events per month for each stock. The frequency of events varies from 0 to 12.8 events per month for each stock. During the study of these events, we found that at the microlevel all events are caused by a temporary liquidity crisis – a moment in trading when one big market order is executed via a large number of trades involving small orders of the opposite direction leading to a leap in price.

A key focus of the paper [26] is the study of the behavior of HFT participants during market shocks. For this purpose, we have identified market movements exceeding 8 standard deviations and 50 basis points in one-minute intervals. The total number of analyzed shocks exceeds 1,000. For the purpose of analysis, shocks accompanied by upward price movements and downward price movements are reviewed separately. Typical shock profiles are provided in Fig. 2.

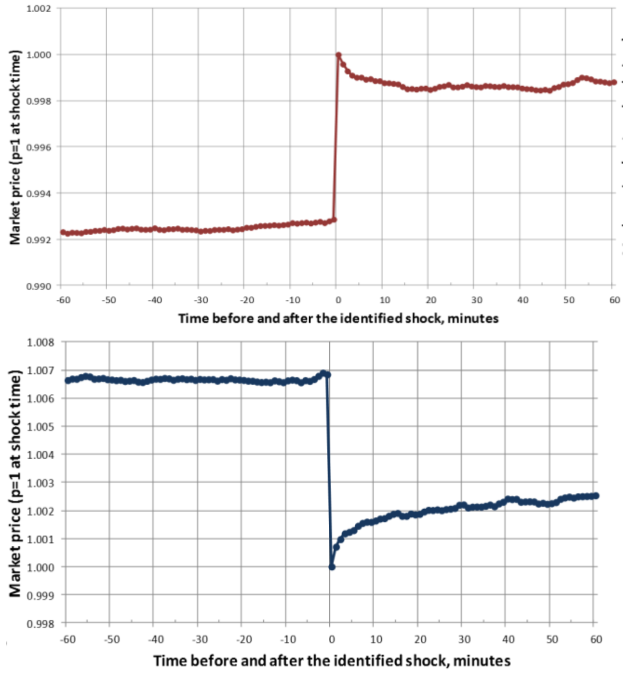

Fig. 2. A typical price shock profile (here and elsewhere an up-shock is illustrated on the top chart, while a down-shock is illustrated on the bottom chart)

For each shock an aggregate trading volume profile is constructed and it has been found that during a shock the volume traded in the market showed a ten-fold increase on average. The analysis shows that a leap in trade imbalance is observed five minutes, on average, before a shock (see Fig.3). In this case, trade imbalance is measured based on a market buy orders to market sell orders ratio:

$I_t = \frac{V_b}{V_s + V_b}$, I_t is trade imbalance at time point t, V_b is aggregate volume of market buy orders, V_s is aggregate volume of market sell orders.

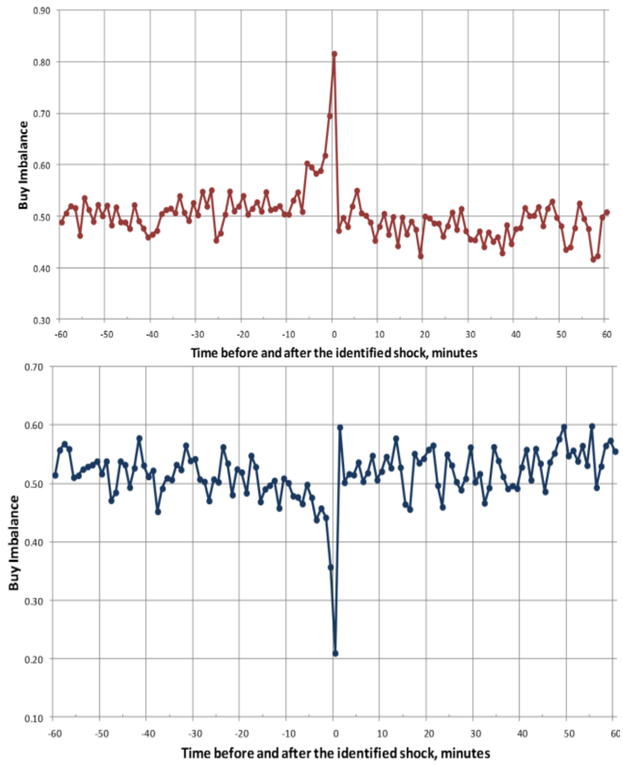

Fig. 3. Aggregate profile of trade imbalance

To analyze the behavior of HFT participants at shock points, a metric describing HTF's aggressive orders for executed trades is considered. During shocks, HFT participants show more aggressive trading and initiate trades in the market.

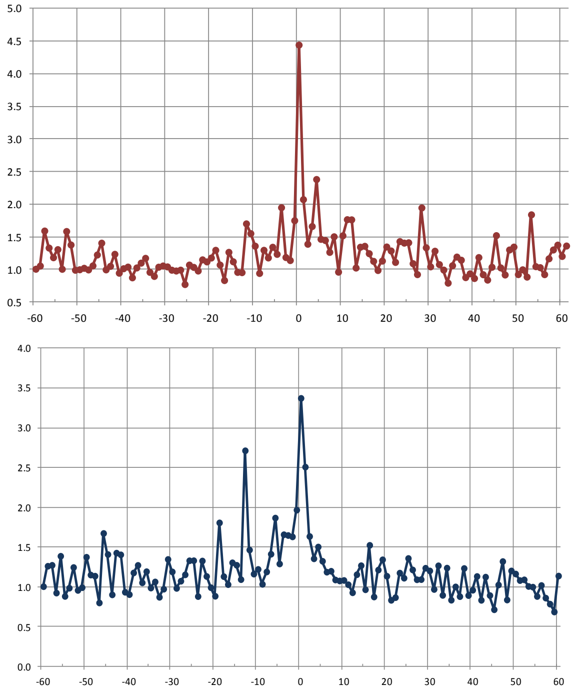

Fig. 4. Aggregate profile of aggressiveness of HFT orders

In the majority of markets, HFTs are present both at the best buy price and best ask price. On a side towards which a shock moves, it can be observed that HFT participants withdraw their orders and enter additional orders on the opposite side (see Fig. 5).

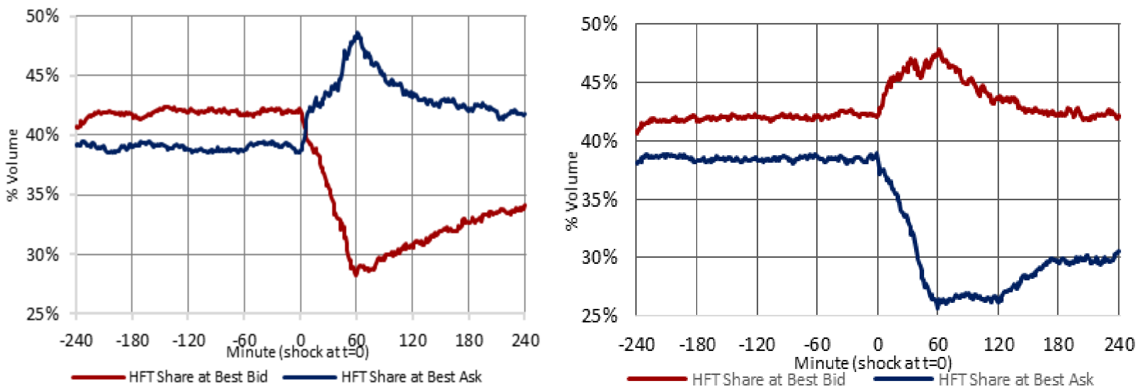

Fig. 5. Aggregate profile of HFT share at best bid and ask prices

Consequently, during shocks HFT participants become more aggressive and supply less liquidity on the shock side.

3.3. Developing a typology of financial market participants

One of the most significant financial market structure developments in recent years is high frequency trading (HFT). Experts say that HFT accounts for the greater part of financial market transactions (for example, according to Tabb Group, HFT accounts for more than 77% of transactions in the UK) and is able to crucially influence the occurrence of systemic instabilities. In paper [39], the following key attributes of HFT algorithms are outlined:

1) Sophisticated high-speed tools. To speed up decision-making, HFT traders use expensive sophisticated tools to track and analyze huge data sets and leverage revealed regularities to make investment decisions in real time. High complexity algorithms and high speed practically exclude a human from decision making.

2) Latency time minimization. There is a direct relationship between the efficiency of trading algorithms and order transfer time from the algorithm to the exchange kernel.

3) Generation of a high amount of messages per day. HFT is often characterized by high amount of messages (order submittals, order updates, order withdrawals, and trade executions), high turnover rates per trading day, high order-to-trade ratios, relatively short average lifetime of orders.

4) Near zero position at the end of trading day. Horizons over which HFT traders hold their positions normally vary from milliseconds to hours.

5) Private firms engaged in proprietary trading.

These algorithms can influence fundamental processes at the level of the market microstructure.

Therefore, identifying of such HFT participants is one of critical tasks. Papers [11; 12; 26] propose different techniques for identifying HFT market participants based on methods for dividing participants into high frequency traders (HFT), long-term investors (LLT), and small participants (SMT). In paper [26], analysis of participants helps identify ten key differentiators explaining over 70% of variations in market participant characteristics. Analysis of these characteristics for one of Asian markets allows identifying about 30 most active accounts having characteristics intrinsic to HFT. This class of accounts is responsible for the generation of more than a half of the aggregate order flow, 75% of all trades (67% of total turnover), and 80% of all price changes.

The ecology of financial market participants is highly dependent on high frequency traders who influence qualitative and quantitative market performance. The paper [22] also shows that the rate of order placement by HFT participants is largely dependent on time of order placement; and a feedback loop strengthens when orders are placed in day time of trading sessions.

In paper [26], to measure the impact of HFT participants on the market, the below vector autoregression (VAR) is used:

$$HFT_{i,t} = a_i + \sum_{k=1}^{n} b_i MQ_{i,t-k} + \sum_{k=1}^{n} c_k HFT_{i,t-k} + \varepsilon_{i,t}$$

$$MQ_{i,t} = \alpha_i + \sum_{k=1}^{n} \beta_i MQ_{i,t-k} + \sum_{k=1}^{n} \gamma_k HFT_{i,t-k} + \in_{i,t}$$

where $HFT_{i,t}$ is aggregate HFT trading volume at time point t, $MQ_{i,t}$ are market variables, n is the number of lags $(n = 1, 2, ..., 6)$.

The following variables are considered as financial market quality variables:

1. Relative spread;
2. Market depth;
3. Mid-point price volatility;
4. Rogers-Satchell volatility;
5. XLM (Xetra Liquidity Measure).

During analysis, we have calculated VAR(p) model for each instrument and each day (over 500 models). Then, for each case we measured Akaike informative criterion (AIC). We have found that the impact of HFTs activity on market characteristics with one-minute lag is insignificant for most cases, except for volatility, which shows positive dependence of HFT trade volume at the previous minute HFT(t-1). In this way, we have found evidence that this leads to an increase in short-term market volatility in the next minute in case of rise in HFT trading volume. We have found no evidence that the financial market liquidity is significantly dependent on HFTs activity (more detailed description of liquidity measures is provided in paper [10]). When it comes to XLM metric, for most cases p-value of lag impact of HFT on this metric is below 5%. The coefficient is statistically significant in 17% of cases. This drives us to a conclusion that HFT market participants do not continuously contribute to the market liquidity; in some cases liquidity drops as HFT trading volume increases.

The results of our studies in this area are used in the PROGNOZ.Timeline, a software tool to analyze financial market microstructure (Fig. 6).

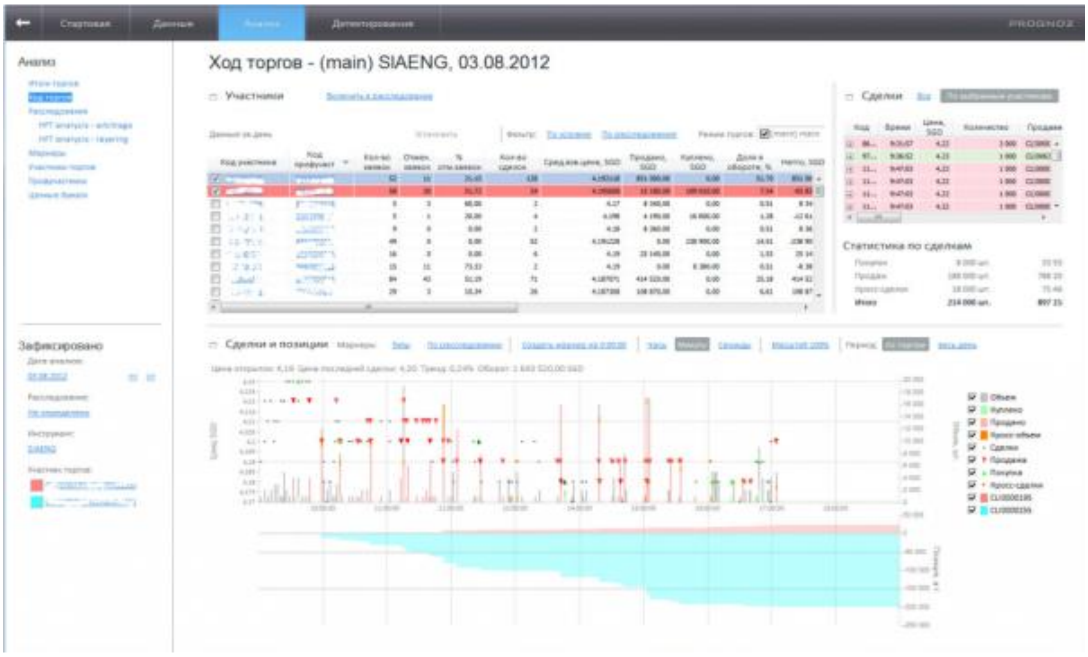

Fig. 6. PROGNOZ. Timeline interface

Access to statistics about trading participants along with powerful modeling and visualization capabilities enables us to perform a wide range of tasks for analysis of financial market microstructure.

3.4. Analysis of financial market regulation consequences

A tick size and lot size of a financial instrument are key parameters used for regulating financial markets. The history of tick size regulation and tick size changes dates back to 1992, when the American Stock Exchange (AMEX) reduced tick size from 1/8 to 1/16th for shares with a price between \$1 and \$5. Tick size reduction consequences are actively discussed in the academic circles. However, there is no consensus whether tick size reduction has positive impact or not.

A tick size is an absolute value and is no good for comparing various instruments or countries or analyzing relationships among variables. For such purposes, it is handier to use a relative minimum price increment (or a relative tick size), which is calculated as follows:

$$relative\ tick\ size = 10000 \cdot \frac{tick\ size}{avg.price}$$

where $relative\ tick\ size$ is relative tick size;

$tick\ size$ is absolute tick size;

$avg.price$ is average price for the calculation period.

The relative tick size is measured in basis points.

The paper [6] analyzes 60 financial instruments with different tick sizes – from very large ones of 72 basis points to very small ones of 0.18 basis points. As tick size reduces, the microstructure of financial instruments changes significantly. It most noticeably manifests in instrument price developments. The paper [8] identifies key properties of order flow and analyzes their relationships with a relative tick size:

- Distribution of order volumes;
- Distribution of order prices;
- Order cancellation rate.

A key characteristic of market order flow is an order size distribution (Fig. 7).

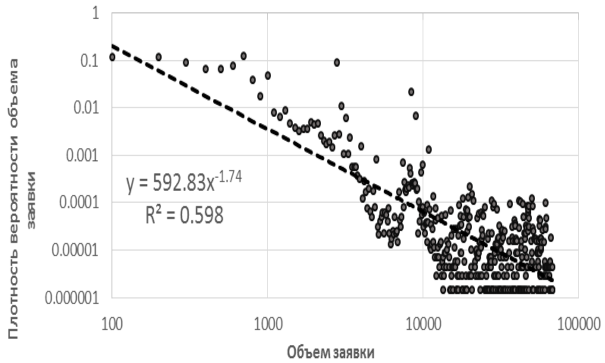

Fig. 7. Distribution of order sizes for Aeroflot common stock

When building distributions for various instruments included into the analyzed sample, we have found that the order size distribution has a power form (see Fig. 8). This means that in case of high tick size larger orders come to the market. In case of small tick size, large orders are broken into smaller ones so that the average order size becomes smaller.

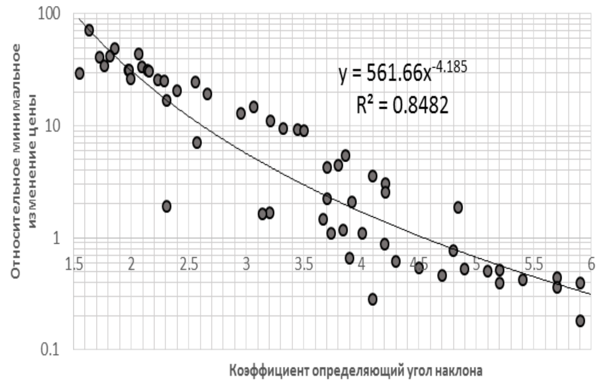

Fig.8. Scatterplot of power-series distribution slope coefficient depending on relative tick size

Prices of incoming market orders depend on the current price of a financial instrument. To compare order price distributions, we need a characteristic that is not dependent on a specific asset price. As such characteristic we use a *price distance*. Let us formulate it as follows:

$price\ distance = (p - p_b) / ticketsize$ for buy orders;

$price\ distance = (p_a - p) / ticketsize$ for sell orders;

where p is price of order a, p_a, p_b are the best bid price and best ask price respectively.

Best price distance in the market is measured by the number of ticks. Having constructed a graph of distribution for this characteristic in paper [9] we found that such distribution is not mixed and is not described by any known distribution used in mathematical statistics (see Fig. 9).

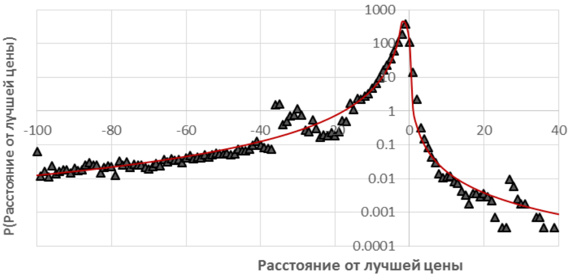

Fig. 9. Distribution of order prices for Aeroflot common stock

For this distribution we have calculated a share of incoming orders at the best price deep into the order book and on the opposite side of the order book (in this way, we divided the distribution into three parts). Having calculated these shares for all instruments in paper [7] we found power-law dependences (see Fig. 10).

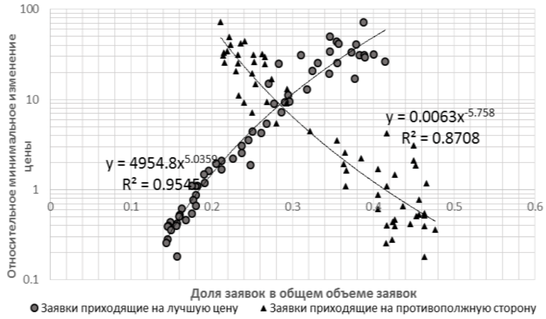

Fig. 10. Scatterplot of a share of incoming orders depending on a relative tick size

For incoming orders at the best price in the market, a slope coefficient is positive. This proves that an increase in tick size leads to an increase in the share of incoming orders at the best price in the market. For orders on the opposite side of the order book, a slope coefficient is negative. In this way, an increase in tick size leads to a decrease in the number of trades.

In paper [7], we have outlined a conditional process for order cancellation. In this process, probability of cancellation, to a large extent, depends on current market conditions with regard to the canceled order. A general formula for conditional probability of order cancellation is as follows:

$$P(\mathrm{C}_i\,|y_i,n_{imb},n_{tot},...) = \mathrm{A}(1-exp^{-K_1 y_i})(1-exp^{-Dn_{tot}})(\mathrm{K}_2\cdot \mathrm{n}_{imb}+\mathrm{b}),$$

where $P(\mathrm{C}_i\,|y_i,n_{imb},n_{tot},...)$ is conditional probability of order cancellation;

y_i – is relative position of the order in the order book;

n_{imb} – is coefficient of order book imbalance;

n_{tot} – is total number of orders in the order book;

A – is a parameter describing maximum probability of order cancellation;

K_1 – is sensitivity of probability of cancellation to relative position of the order in the order book;

D – is sensitivity of probability of cancellation to total number of orders;

K_2 – is sensitivity of probability of cancellation to order book imbalance;

b – is adjustment to coefficient of imbalance.

Tick size impact analysis shows that there is a dependence on coefficient K_1 (see Fig. 11). The coefficient K_1 characterizes order price sensitivity to the current market conditions. Order price sensitivity to the current price position in the market decreases as tick size goes down.

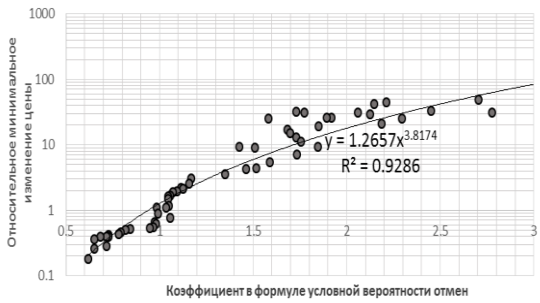

Fig. 11. Scatterplot of dependence of coefficient of conditional probability of cancellation on relative tick size

In paper [7], we have formulated a method that enables them to take into account consequences of tick size changes in order flow properties. The identified dependencies can be written as follows:

$$\begin{cases} K_1 = 0{,}94 \cdot TS^{\,0{,}2619} \\ \alpha_{volume} = 4{,}512 \cdot TS^{\,-0{,}238} \\ Q_{same} = 0{,}185 \cdot TS^{\,0{,}198} \\ Q_{opposite} = 0{,}416 \cdot TS^{\,-0{,}178} \\ Q_{best} = 1 - 0{,}416 \cdot TS^{\,-0{,}178} - 0{,}185 \cdot TS^{\,0{,}198} \end{cases}$$

where K_1 – is sensitivity of probability of order cancellation to relative position of the order in the order book;

α_{volume} – is slope of power law for order size distribution;

Q_{same} – is probability of order booking in the order book (for buy orders, the price is lower the best ask price; for sell orders, the price is higher the best bid price);

$Q_{opposite}$ – is probability of order booking on the opposite side of the order book (for buy orders, the price is higher the best ask price; for sell orders, the price is lower the best bid price);

Q_{best} – is probability of order booking at the best price (for buy orders, the price is equal to the best ask price; for sell orders, the price is equal to the best bid price);

TS is relative tick size in the market.

Laying down the rules regulating a tick size remains a live issue today. Stock exchanges and financial market regulators continue to change the rules for defining a tick size. During the study of this topic, we have found that the tick size reduction leads to:

- Decrease in average order size in the market;
- Increase in the number of trades in the market;
- Decrease in the number of orders at the best price in the market;
- Increase in total probability of order cancellation;
- Higher sensitivity of probability of cancellation to market situation.

These conclusions are in good agreement with the results obtained during the real market regulation. The proposed method can be used for predicting the impact of tick size changes on the microstructure of financial markets.

3.5. Simulation of financial markets

Simulation of complex systems has been successfully applied across a variety of research and industry sectors. The first agent-based simulation model of financial market—the Santa Fe Artificial Stock Market (ASM) model—was built by a group of researchers at the Santa Fe Institute in the early 90ies. Being mainly experimental, the model involved a number of artificially intelligent agents, or traders, deciding how much to invest in a risky asset and how much to invest instead in a risk-free asset. Each trader does this by generating a demand for a financial asset while the asset price moves in response to an imbalance between its demand and supply. The ASM model has inspired the emergence of various similar models demonstrating different assumptions. However, such models are not flawless as they rely on a great number of assumptions and non-empirical parameters. Backtesting for such a class of models is reduced to selecting parameters which describe empirical data in the most comprehensive way.

Due to technological advances, researchers are now provided with much more detailed trading data. The most complete information can be derived from transactional data containing entries of all market orders and trades. Within our problem, we have developed an approach to build simulation models of a stock market microstructure, which–unlike other comparable models–can factor in tick size changes made by financial market regulators as well as limitations or preferences for a specific class of market participants. Our stock market microstructure model is based on the Mike-Farmer zero intelligence model designed under the guidance of Prof. J. D. Farmer. As demonstrated in [8], the Mike-Farmer model has a number of limitations affecting the quality and accuracy of the simulation model; it cannot also factor in external influences such as regulator-imposed constraints. Our simulation model draws on the Mike-Farmer model while overcoming its drawbacks. The resulting model includes two types of agents such as noise traders and high-frequency traders (Fig. 12).

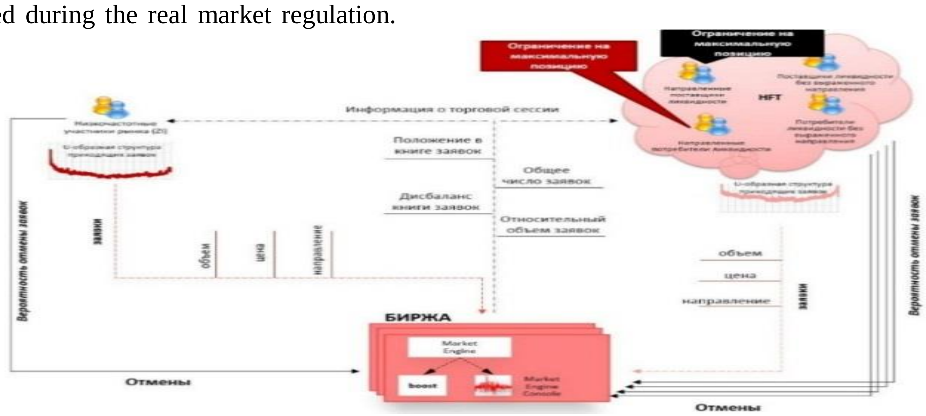

Fig. 12. Simulation model of stock market microstructure

High-frequency traders fall into four types here: directional liquidity providers (intermarket arbitrageurs), directional liquidity consumers (smart order routing, or SOR), non-directional liquidity providers (market makers), and non-directional liquidity consumers (statistical arbitrageurs). Each trader enters and withdraws orders in the market.

Each entered order has a direction, a size, and a price. To simulate the order direction, we have used the methods developed by Prof. F. Lillo for reproducing a long-memory order flow. Prof. Lillo suggested modeling the order size as a power law distribution that allows for simulating large orders coming into the market along with significant microstructural changes that result from this inflow. To model the order price, we considered its distribution from the best market prices. [9] is the first research publication to suggest splitting this distribution into three parts in order to factor in the tick size influence (Fig. 13).

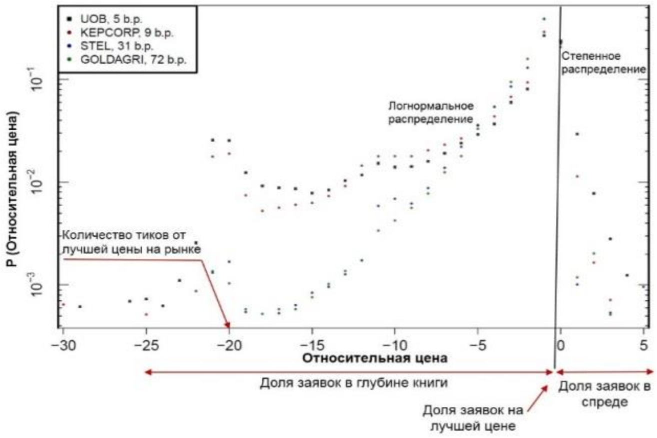

Fig.13. Distribution of the order price from the best market prices

The order distribution parameters and the u-shape pattern of the order flow intensity are estimated for each agent. Each trader cancels orders with a specified probability. Being conditional, this probability depends on the current market situation. We have identified the following most significant parameters that can influence the order cancellation process in the financial market:

- Position of the order in the order book relative to the current market prices, or y_i;
- Total number of orders in the order book, or n_{tot};
- Imbalance of orders in the order book, or n_{imb};
- Relative size of orders, or v_{rel}.

The results for each aforementioned characteristic are combined to construct a conditional probability function describing the cancellation process (Fig. 14).

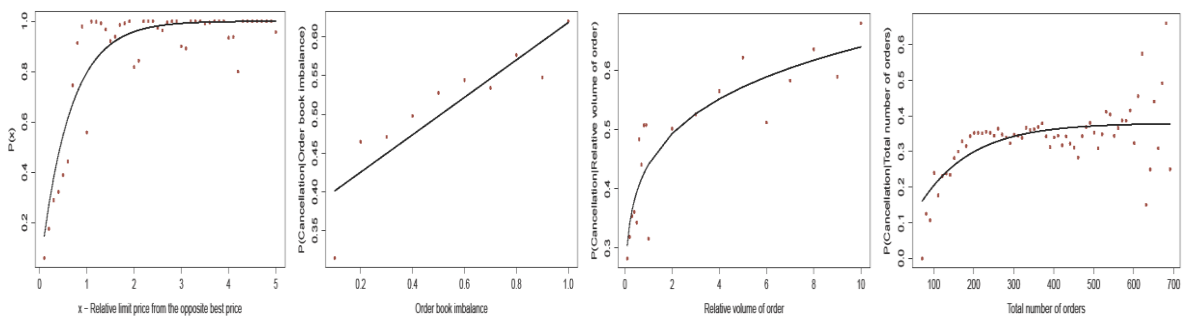

Fig. 14. Conditional probability function describing the cancellation process depending on the distribution of the order price from the best market prices (top-left), relative order size (bottom-left), total number of orders (bottom-right), order book imbalance (top-right)

The order cancellation process is not explicitly time-dependent however it depends on the state of the order book. The order cancellation process parameters K_1 and b are dependent from the tick size for a particular financial instrument. Order cancellation functions have also been analyzed for a number of traders identifying components that are not significant in the order cancellation process. For example, the order book imbalance is not significant for high-frequency traders who are directional liquidity consumers. To enable the actual implementation of our simulation model, we have designed a dedicated software suite modeling the stock exchange, its operations, main elements and their interaction. The model kernel runs on C++.

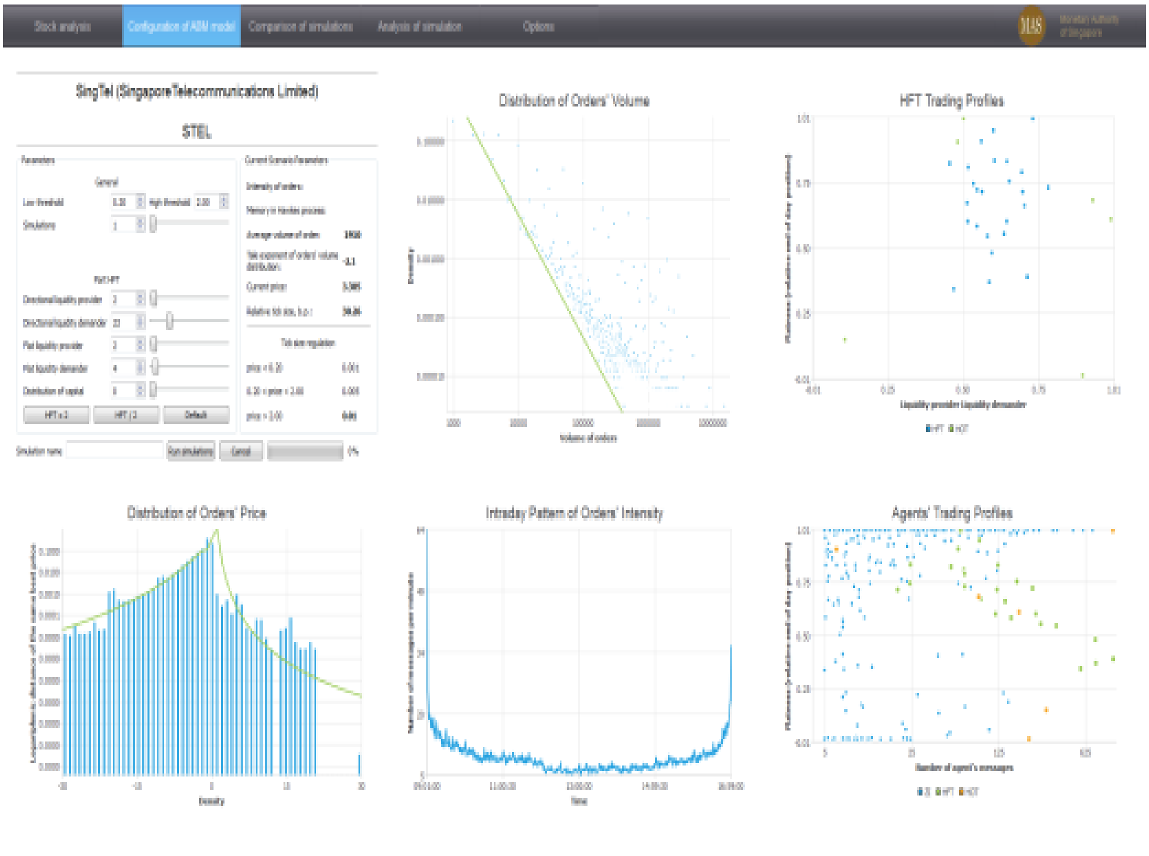

Fig. 15. Simulation modeling system, configuration module of the simulation model

Our simulation model enables users to make scenario changes with regard to preferences and limitations for particular market participants as well as to analyze market shifts triggered by tick size changes.

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ДИНАМИЧЕСКИЕ МОДЕЛИ ЭКОНОМИКИ: ТЕОРИЯ, ПРИЛОЖЕНИЯ, ПРОГРАММНАЯ РЕАЛИЗАЦИЯ

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Дается обзор теоретических и прикладных результатов, полученных в рамках научной школы кафедры «Информационные системы и математические методы в экономике». Обзор охватывает период 2008–2015 гг. В основе теоретических результатов лежат основные положения современной теории функционально-дифференциальных уравнений, разработанной участниками известного Пермского семинара по функционально-дифференциальным уравнениям под руководством профессора Н.В. Азбелева (1922–2006). В центре внимания находятся задачи прогнозирования, краевые задачи (задачи достижимости), задачи управления и задачи устойчивости для динамических моделей, учитывающих эффекты последействия и возможность импульсных воздействий (шоков). Для упомянутых задач получены признаки их разрешимости, предлагаются методы построения программных управлений и соответствующих им траекторий, разработаны схемы и алгоритмы исследования на основе доказательного вычислительного эксперимента, включающие алгоритмы коррекции исследуемых задач в случае обнаружения их противоречивости. Результаты прикладных разработок используют достижения теории и представляют собой комплекс программных средств для исследования на разрешимость и решения реальных задач прогнозирования, достижимости, управления и устойчивости для моделей социально-экономического развития субъектов Российской Федерации и российской экономики в целом.

Ключевые слова: модели экономической динамики, задачи прогнозирования, краевые задачи, задачи управления, информационно-аналитические системы, системы поддержки принятия решений, прогнозно-аналитические системы, бизнес-аналитика, финансовое моделирование, финансовые рынки.

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